

Quantum Gravity: Mixed States from Diffeomorphism Anomalies

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Abstract

In a previous paper, we discussed simple examples like particle on a circle and molecules to argue that mixed states can arise from anomalous symmetries. This idea was applied to the breakdown (anomaly) of color $SU(3)$ in the presence of non-abelian monopoles. Such mixed states create entropy as well.

In this article, we extend these ideas to the topological geons of Friedman and Sorkin in quantum gravity. The “large diffeos” or mapping class groups can become anomalous in their quantum theory as we show. One way to eliminate these anomalies is to use mixed states, thereby creating entropy. These ideas may have something to do with black hole entropy as we speculate.

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1 Introduction

Diffeomorphisms of space-time play the role of gauge transformations in gravitational theories. Just as gauge invariance is basic in gauge theories, so too is diffeomorphism (diffeo) invariance in gravity theories.

Diffeos can become anomalous on quantization of gravity models. If that happens, these models cannot serve as descriptions of quantum gravitating systems.

There have been several investigations of diffeo anomalies in models of quantum gravity with matter in the past. For example, Alvarez-Gaumé and Witten showed the absence of these anomalies in string theories [1].

But all these studies have dealt with “small diffeos” or the identity component of the diffeo group. Some references that deals with anomalies associated with “large diffeos” are [2, 3]. For “large gauge” anomalies, see [4–7].

We here consider asymptotically flat space-times and focus on the diffeo group D^∞ of the spatial slice M^d of dimension d which keeps the flat metric and a frame at ∞ fixed. The identity component D_0^∞ of D^∞ is what is required to act trivially on quantum states by the diffeomorphism constraints. The group of “large diffeos” D^∞/D_0^∞ is called the mapping class group of M^d . It can act non-trivially on quantum states, but observables commute with it. It is a discrete group.

If M^d is \mathbb{R}^d , then $D^\infty/D_0^\infty = \{e\}$. So we need more interesting models of M^d for work on large spatial diffeos. They are provided by the geon manifolds of Friedman and Sorkin [8–10]. We study such manifolds for $d = 2$ and $d = 3$, and show that they can become anomalous.

Our approach towards this demonstration is based on the work of Esteve [11, 11, 12]. It is as follows. We consider the Dirac operator or the Laplacian of a matter field on the manifold M^d . The eigenmodes of these operators enter the mode expansion of the matter field and its second quantization. The Dirac operator or the Laplacian must be self-adjoint in order to have a complete set of orthonormal eigenstates and real eigenvalues so that it can be used in the above mode expansion.

Now the proper definition of these operators as self-adjoint operators involves the choice of a domain for them: there can be several inequivalent choices leading to different physics. We then show cases where these domains are changed by D^∞/D_0^∞ proving that they are anomalous [11]. These anomalies are similar to the parity and time-reversal anomalies for certain domains of the Laplacian for a particle on a circle or for several different molecules [5, 13].

The large diffeos leave the Dirac operator and the Laplacian invariant if the metric is also transformed. Still the domain can be changed by “large

diffeos” so that “a classical diffeo symmetry becomes anomalous in quantum theory”.

There is another approach to this question of diffeos and their compatibility with domains. It is based on Witten’s proof of positive energy theorem [14]. Witten uses a Dirac operator on a spatial slice which includes the influence of both gravity and matter. Its definition involves the proper choice of domains to guarantee ellipticity [14,15]. If diffeos change this domain, then they are surely anomalous.

But this line of argument requires more articulation as we will see.

In section 2, we give a presentation of simple geon manifolds M^d for $d = 2$ and $d = 3$ in a form convenient for the present work.

In section 3, we discuss domains for Dirac operators where they are self-adjoint. We also discuss how we can recover spatial topology from these domains, thereby making progress in the problem of the confused quantum baby [16].

In section 4, we study the action of the diffeo group on domains and show how it can change them. Impure states which eliminate such anomalies are also constructed.

In section 5, we look at the Dirac operator used by Witten in his proof of the positive energy theorem, but this time on geon manifolds. It includes gravity, indeed the ADM Hamiltonian can be written using it. Its proper treatment certainly involves a domain choice. Just as previously, large diffeos change several of these domains.

In the concluding section 6, we speculate on the entropy of the mixed states associated with large diffeos and why it may have something to do with black hole entropy.

2 A Presentation of Geon Manifolds

If M_1 and M_2 are two manifolds of dimension d , their connected sum $\#$ is defined as follows [17]. Remove balls B_i , with $i = 1, 2$, from M_i . Then $M_i \setminus B_i$ are manifolds with spheres S^{d-1} as boundaries. Identify these spheres to obtain $M_1 \# M_2$. If M_i are oriented, and this identification is done with orientation reversal, then $M_1 \# M_2$ is also oriented.

There is a class of closed (that is, compact and boundaryless) manifolds for $d = 2$ and $d = 3$ which are called primes. *All* closed (that is, compact and boundaryless) manifolds are connected sums of primes.

For $d = 2$, there is only one orientable prime, namely the two-torus T^2 . For $d = 3$, there are an infinity of them, one being the three-torus T^3 .

In this paper, we will focus on $\mathbb{R}^d \# T^d$, with $d = 2, 3$, because they are

relatively simple and also illustrate our ideas. They are asymptotically flat: all asymptotically flat two- and three-dimensional manifolds with one asymptotic region are obtained by attaching a finite number of primes P_α of dimension d to \mathbb{R}^d , that is, they are $\mathbb{R}^d \# P_1 \# P_2 \# \dots \# P_k$.

Now, there is an elegant way to present $\mathbb{R}^d \# P$ if P is a prime. We take out a ball B^d from \mathbb{R}^d to obtain $\mathbb{R}^d \setminus B^d$ and then make suitable identifications of points on the boundary $\partial(\mathbb{R}^d \setminus B^d)$ of $\mathbb{R}^d \setminus B^d$.

Let us show this for $\mathbb{R}^2 \# T^2$. A square with its interior is as good as a two-ball (disk), as the former can be deformed to the latter. So we remove such a square, call it B^2 , from \mathbb{R}^2 . Then we identify opposite sides of $\partial(\mathbb{R}^2 \setminus B^2)$ to obtain $\mathbb{R}^2 \# T^2$. Figure 1 displays this construction.

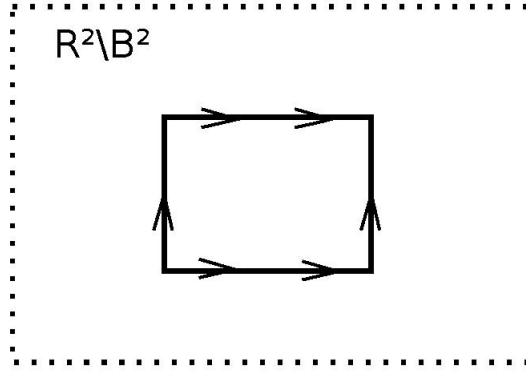


Figure 1: A presentation of $\mathbb{R}^2 \# T^2$

A similar method works for $\mathbb{R}^3 \# T^3$: we remove a cube from \mathbb{R}^3 and identify opposite faces.

We can think of the empty square and cube with their identifications in \mathbb{R}^d , with $d = 2, 3$, as the spatial location of geons where we carved holes in \mathbb{R}^d are where the geons have been attached.

This presentation is very convenient since we can use the flat metrics away from the holes to treat the Dirac operator and Laplacian on $\mathbb{R}^d \setminus B^d$, B^d being the square and the cube for $d = 2, 3$. The identifications at the boundaries can be incorporated in the operator domains. That gives us relatively simple models to deal with. But we do not expect the conclusion to be sensitive to our choice of models.

3 On Domains of Operators

3.1 $\mathbb{R}^2 \# T^2$

The Dirac operator we deal with is

$$i\mathcal{D} = i\gamma \cdot \partial \tag{1}$$

$$\gamma^a = \sigma^a, \quad a = 1, 2 \tag{2}$$

$$\partial_a = \frac{\partial}{\partial x^a}, \tag{3}$$

where (x^1, x^2) are the Cartesian coordinates on $\mathbb{R}^2 \setminus B^2$ and σ^a are Pauli matrices. Gauge fields can be included in (1), but we do not do so here for simplicity. See below for further comments on gauge fields.

The operator $i\mathcal{D}$ is defined on $\mathbb{R}^2 \setminus B^2$ which has the square as boundary.

The boundary conditions on $\partial(\mathbb{R}^2 \setminus B^2)$ define the domain \mathcal{D} of $i\mathcal{D}$. It must be chosen so that $i\mathcal{D}$ is self-adjoint. For technical details about the latter, see [18–20].

The topology of the underlying manifold is also encoded in \mathcal{D} . This aspect is of importance for this work. Thus if the underlying manifold has to have the topology of T^2 , then \mathcal{D} must be a representation space for continuous functions on T^2 . That is to say, it must be a module for the \mathbb{C}^* -algebra $\mathcal{C}^0(T^2)$. Then we can recover T^2 as a topological space by the Gelfand-Naimark theorem [21].

If we want a more refined statement on T^2 and recover also its differential structure, that can be done by requiring that \mathcal{D} is a module for the algebra of once-differentiable functions $\mathcal{C}^1(T^2)$ on T^2 . We can keep on going in this manner and require that \mathcal{D} is a module for $\mathcal{C}^\infty(T^2)$, the algebra of infinitely differentiable functions on T^2 .

We will see that such a domain exists. It is one where \mathcal{D}^k is “essentially self-adjoint” for all k .

Let us see how to find such a domain. For purposes of describing it, let us choose an origin of \mathbb{R}^2 in the middle of the square, and give its boundaries the coordinates

$$\left\{ \left(\pm \frac{1}{2}, y \right) : -\frac{1}{2} \leq y \leq \frac{1}{2} \right\}, \quad \text{and} \tag{4}$$

$$\left\{ \left(x, \pm \frac{1}{2} \right) : -\frac{1}{2} \leq x \leq \frac{1}{2} \right\}. \tag{5}$$

See Figure 2.

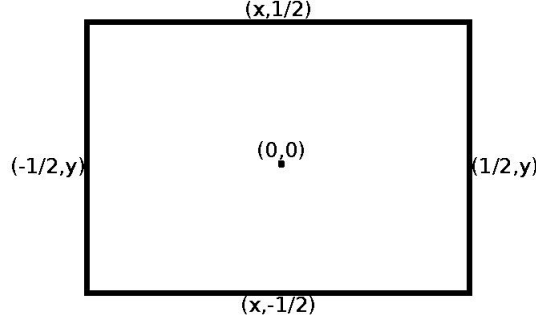


Figure 2: The coordinates for $\partial(\mathbb{R}^2 \setminus B^2)$

Then in \mathcal{D} , call it $\mathcal{D}^{(0)}$, we allow only smooth L^2 -functions in $\mathbb{R}^2 \setminus B^2$ which fulfill the boundary conditions

$$\mathcal{D}^{(0)} = \left\{ \psi : \psi(x, \frac{1}{2}) = e^{i\theta_1} \psi(x, -\frac{1}{2}), \psi(\frac{1}{2}, y) = e^{i\theta_2} \psi(-\frac{1}{2}, y), \theta_i \in \mathbb{R} \right\} \quad (6)$$

Functions χ on $\mathcal{C}^0(T^2)$ are periodic on the square: $\chi(x, \frac{1}{2}) = \chi(x, -\frac{1}{2})$ and $\chi(\frac{1}{2}, y) = \chi(-\frac{1}{2}, y)$. This is in accordance with Figure (1) for $e^{i\theta_i} = 1$. One sees that if ψ fulfills (6), so does $\chi\psi$. As we have taken $\mathcal{D}^{(0)}$ to consist of smooth functions, we should take χ 's also to be smooth. Then the completion of the algebra of such χ 's in the sup norm gives us back $\mathcal{C}^0(T^2)$.

It is not difficult to prove that \not{D} is essentially self-adjoint on $\mathcal{D}^{(0)}$

But \not{D}^2 is not self-adjoint for domain $\mathcal{D}^{(0)}$. For that, we require the domain

$$\mathcal{D}^{(1)} = \left\{ \psi \in \mathcal{D}^{(0)} : \frac{\partial \psi}{\partial y}(x, \frac{1}{2}) = e^{i\theta_1} \frac{\partial \psi}{\partial y}(x, -\frac{1}{2}), \quad (7)$$

$$\frac{\partial \psi}{\partial x}(\frac{1}{2}, y) = e^{i\theta_2} \frac{\partial \psi}{\partial x}(-\frac{1}{2}, y) \right\} \quad (8)$$

That is, both ψ and its normal derivatives to the boundary must be quasi-periodic.

For self-adjointness of \not{D}^N , we similarly require a domain

$$\mathcal{D}^{(N-1)} = \left\{ \psi \in \mathcal{D}^{(0)} : \frac{\partial^k \psi}{\partial y^k}(x, \frac{1}{2}) = e^{i\theta_1} \frac{\partial^k \psi}{\partial y^k}(x, -\frac{1}{2}), \quad (9)$$

$$\frac{\partial^k \psi}{\partial x^k}(\frac{1}{2}, y) = e^{i\theta_2} \frac{\partial^k \psi}{\partial x^k}(-\frac{1}{2}, y), \quad \forall k \leq N-1 \right\}. \quad (10)$$

On $\mathcal{D}^{(N-1)}$, \not{D}^N is self-adjoint whereas \not{D}^m for $m < N$ are only essentially self-adjoint. But that is enough for us.

For a domain $\mathcal{D}^{(\infty)}$ which is a module for $\mathcal{C}^\infty(T^2)$, we should take the intersection of all $\mathcal{D}^{(N)}$, that is,

$$\mathcal{D}^{(\infty)} = \bigcap_N \mathcal{D}^{(N)}. \quad (11)$$

On $\mathcal{D}^{(\infty)}$, \not{D}^N are essentially self-adjoint for all N .

It is easy to see that $\mathcal{D}^{(\infty)}$ exists. For that, consider smooth functions in $\mathcal{D}^{(0)}$ which are constant on a “collar neighborhood” of $\partial(\mathbb{R}^2 \setminus B^2)$. This neighborhood is shaded in Figure (3). These functions belong to $\mathcal{D}^{(\infty)}$.

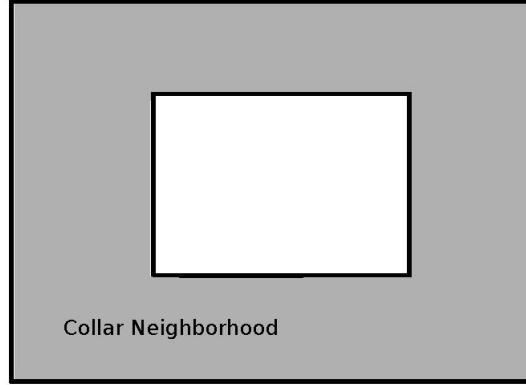


Figure 3: The collar neighborhood of $\partial(\mathbb{R}^2 \setminus B^2)$

3.2 Absorbing Boundary Conditions into Gauge Fields

The boundary data of (6) can be described in terms of a $U(1)$ -valued field U on $\partial(\mathbb{R}^2 \setminus B^2)$ as we now show. The definition of U is

$$U(x, +\frac{1}{2}) = e^{i\theta_1} U(x, -\frac{1}{2}), \quad (12)$$

$$U(+\frac{1}{2}, y) = e^{i\theta_2} U(-\frac{1}{2}, y), \quad (13)$$

with $|x|, |y| \leq 1/2$.

If the domain $\mathcal{D}^{(0)}$ is called $\mathcal{D}_V^{(0)}$, then the fields in $\mathcal{D}_1^{(0)}$ ¹ have $e^{i\theta_i} = 1$. Also if

$$\chi_1 \in \mathcal{D}_1^{(0)}, \quad (14)$$

then

$$U\chi_1|_{\partial(\mathbb{R}^2 \setminus B^2)} \quad (15)$$

behaves on the boundary according to (6).

We can extend U to all of $\mathbb{R}^2 \setminus B^2$ easily. For example, we can cover this space by squares and declare U to be constant on radial lines as in Figure 4. Across each square, it is quasi-periodic as in the square of Figure 2.

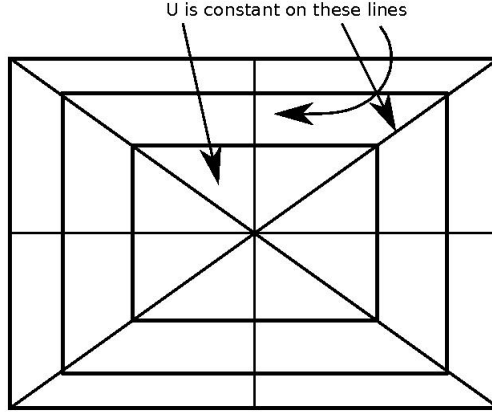


Figure 4: Extension of U to $\mathbb{R}^2 \setminus B^2$ from $\partial(\mathbb{R}^2 \setminus B^2)$.

But this extended U is not differentiable at the corners of the squares.

But there is a different domain with a different V at the boundary where this problem can be overcome. Instead of a square for B^2 , let us choose a disk with boundary as a smooth circle S^1 as in Figure 5.

For such V explicitly defined in Figure 5, we can define a new domain

$$\mathcal{D}_V^{(0)} = \{\psi : \psi|_{\partial(\mathbb{R}^2 \setminus B^2)} = V\chi_1|_{\partial(\mathbb{R}^2 \setminus B^2)}\} \quad (16)$$

where $\chi_1 \in \mathcal{D}_1^{(0)}$.

The operator $i\mathcal{D}$ is (essentially) self-adjoint on $\mathcal{D}_V^{(0)}$. Also $\mathcal{D}_V^{(0)}$ is a module for $\mathcal{C}^\infty(T^2)$ just as we want.

As V is smooth on $\partial(\mathbb{R}^2 \setminus B^2)$, there is no problem in extending it as a smooth $U(1)$ -valued function V on all of $\mathbb{R}^2 \setminus B^2$. We may want to require

¹Here $\mathbb{1}$ has the constant value 1 on $\partial(\mathbb{R}^2 \setminus B^2)$.

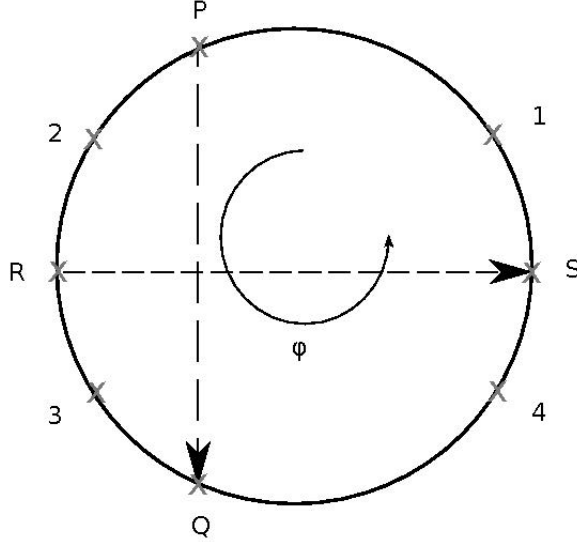


Figure 5: A smooth V on S^1 : $V(S) = e^{i\theta_1(\varphi(P))}V(R)$ ($V(P) = e^{i\theta_2(\varphi(Q))}V(Q)$), where $P \in [1, 2]$ and $Q \in [3, 4]$ ($R \in [2, 3]$, $S \in [4, 1]$) are vertically (horizontally) opposite points. The functions $e^{i\theta_i(\varphi)}$, with $i = 1, 2$, are smooth functions on S^1 , with φ being the angular coordinate on S^1 . At point $S \in S^1$, this coordinate is $\varphi(S)$. Similarly at point P , it is $\varphi(P)$, and so on.

that V approaches the same constant value V_∞ as $|x| \rightarrow \infty$ in any direction., but that too is easily arranged.

Now if $i\not{D}$ has domain $\mathcal{D}_U^{(0)}$, then

$$V^{-1}i\not{D}V \quad (17)$$

has domain $\mathcal{D}_1^{(0)}$ as $\mathcal{D}_U^{(0)}$ is determined only by the boundary value of V .

The operator (17) is a Dirac operator with a flat connection $V^{-1}(DV)$, that is,

$$V^{-1}i\not{D}V = i\gamma \cdot (D + V^{-1}(DV)). \quad (18)$$

Thus for such a U , we can work with a fixed domain $\mathcal{D}_1^{(0)}$ and a Dirac operator with a connection.

We have earlier discussed this transformation of boundary conditions to connections in the simple case of a particle on a circle [22].

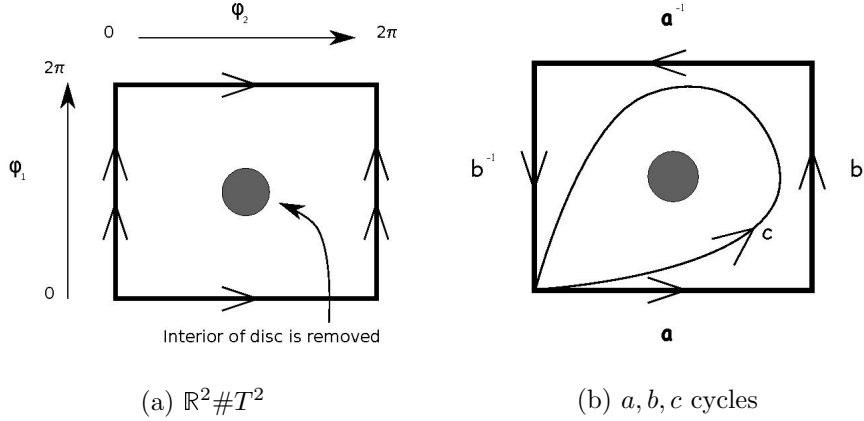


Figure 6: (a) A new presentation of $\mathbb{R}^2 \# T^2$; (b) The a, b, c cycles of $\pi_1(\mathbb{R}^2 \# T^2)$.

3.3 The Role of the π_1 -Group

The above discussion can be framed differently [17]. The manifold $\mathbb{R}^2 \# T^2$ is multiply connected. Its fundamental group $\pi_1(\mathbb{R}^2 \# T^2)$ is actually nonabelian with presentation

$$\pi_1(\mathbb{R}^2 \# T^2) = \langle a, b, c : c = aba^{-1}b^{-1}, ac = ca, bc = cb \rangle. \quad (19)$$

See the paper [10] for a proof. For T^2 , $c = e$, the identity. But here c generates the non-trivial center of the fundamental group. It comes from the fact that a loop “circling completely the geon” cannot be deformed to a point. Figure 6 shows the a, b and c cycles.

On multiply connected spaces such as $\mathbb{R}^2 \# T^2$, there is a vector bundle with a flat connection which implements a unitary irreducible representation (UIRR) of $\pi_1(\mathbb{R} \# T^2)$. Its sections then go to define domains of operators like $i\mathcal{D}$ [17].

In the example above, we chose a UIRR wherein

$$a \rightarrow e^{i\theta_1} \mathbb{1}, \quad (20)$$

$$b \rightarrow e^{i\theta_2} \mathbb{1}, \quad (21)$$

so that

$$c \rightarrow \mathbb{1}, \quad (22)$$

and the representation is abelian.

There are non-abelian representations as well. A simple example is provided by the “rational torus”. In the $N \times N$ representation of $\pi_1(\mathbb{R}^2 \# T^2)$ by

the rational torus, a, b are represented by “clock”- and “shift”- operators U_1 and U_2 , that is,

$$a \rightarrow U_1, \quad (23)$$

$$b \rightarrow U_2, \quad (24)$$

while c is a root of unity

$$c = e^{i\frac{2\pi}{N}} \mathbb{1}_{N \times N}. \quad (25)$$

We also impose the conditions

$$U_1^N = e^{i\theta_1} \mathbb{1}, \quad (26)$$

$$U_2^N = e^{i\theta_2} \mathbb{1}. \quad (27)$$

As a side remark, we observe that one can also represent c by $e^{i2\pi\frac{p}{N}} \mathbb{1}_{N \times N}$ where p is a fixed integer in $[1, 2, \dots, N-1]$. If this differs from (25), then it is a new representation.

How do we adopt this representation to define domains for Dirac operators?

That is really easy. For $\mathbb{R}^2 \# T^2$ we let $i\mathcal{D}$ to act on \mathbb{C}^N -valued spinors ψ so that it has spinor and “flavor” indices $a(\in [1, 2])$ and $\rho(\in [1, 2, \dots, N])$, respectively. That is, $\psi = (\psi^{a,\rho})$. Then on $\partial(\mathbb{R}^2 \setminus B^2)$, we replace $e^{i\theta_i}$ in (6) by U_i , where the matrices U_i act on the flavor index ρ .

3.4 Generalizations

In three spatial dimensions as well, $\mathbb{R}^3 \# P$, where P is an oriented prime, can be represented by carving out a ball B^3 from \mathbb{R}^3 and making appropriate identifications on the boundary. For example, for $\mathbb{R}^3 \# T^3$, taking the ball to be a cube, we identify the opposite faces of its boundary to obtain $\mathbb{R}^3 \# T^3$ as in Figure (7).

The fundamental group $\pi_1(\mathbb{R}^3 \# T^3)$ is nonabelian. We do not need its details here. It is sufficient to know that it has vector bundles with flat connections associated to non-abelian UIRR’s of this group.

4 On How Diffeos Can Change Domains

The reason why this can happen is as follows. Diffeos of a manifold M which leave a point P (and a frame at P) of M fixed act on $\pi_1(M)$. The reason why we require the diffeos to leave a point P fixed is that elements of $\pi_1(M)$ are equivalence classes of loops starting and ending at a *fixed* point, which

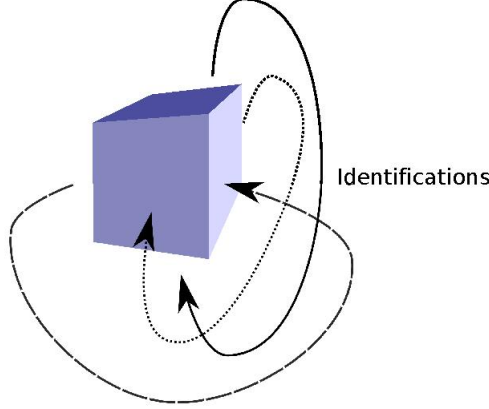


Figure 7: Opposite faces of the cube should be identified. The figure shows the identified top- and bottom-faces. Similar identifications must be made on side faces.

we can take to be P . One says that the loops are thus based at P . Diffeos leaving P fixed map loops based at P to other loops based at P , and hence they act on $\pi_1(M)$ ².

For asymptotically flat spaces such as those we consider, we take P to be the “point at ∞ ”. The diffeos of D^∞ become trivial at ∞ , and so leave P fixed. It follows that D^∞ acts on $\pi_1(\mathbb{R}^d \# P)$.

But D_0^∞ acts trivially on $\pi_1(\mathbb{R}^d \# P)$. For if $d \in D_0^\infty$, there is a curve $d_t \in D_0^\infty$ such that $d_1^\infty = d$, $d_0^\infty = e$. The action of d on a loop l based at P can thus be continuously deformed to the action of identity. Now the loops $d_t l = \langle d_t x(t) : x(0) = x(1) = P \rangle$ are all based at P and as t decreases from 1 to 0, it deforms $d_1 l = d l$ to l without ever changing the base point P of the intermediate loops. Thus $d l$ is homotopic to l and D_0^∞ acts trivially on $\pi_1(\mathbb{R}^d \# P)$.

But “large diffeos” can act nontrivially on $\pi_1(\mathbb{R}^d \# P)$. This action is an automorphism as well. Also since D_0^∞ acts trivially, the effective action is just that of D^∞/D_0^∞ , the mapping class group.

For $d \in D^\infty$, let $\tau(d)$ denote this automorphism on $\pi_1(\mathbb{R}^d \# P)$. If ρ is a representation of $\pi_1(\mathbb{R}^d \# P)$, set

$$[\tilde{\tau}(d)\rho](g) = \rho(\tau(d)^{-1}g), \quad (28)$$

²Changing the base point on a pathwise connected manifold gives the “same” (that is, isomorphic) group $\pi_1(M)$.

for $g \in \pi_1(\mathbb{R}^d \# P)$. Then $\tilde{\tau}(d)\rho$ is a representation too, which may or may not be equivalent to ρ . If the two representations are inequivalent, we write

$$\tilde{\tau}(d)\rho \neq \rho. \quad (29)$$

Since ρ fixes the domain, in the case (29), we can be sure that the diffeo d is domain-changing and hence anomalous.

A transformation $\tilde{\tau}(d)$ can be anomalous even if $\tilde{\tau}(d)\rho = \rho$. We will come back to this point later.

4.1 The Diffeo Anomaly for $\mathbb{R}^2 \# T^2$

It is well-known that the mapping class group of T^2 is $SL(2, \mathbb{Z})$. See [23]. This is not quite the same as D^∞/D_0^∞ of $(\mathbb{R}^2 \# T^2)$. The latter is the Steinberg group $St(2, \mathbb{Z})$ [10].

We can describe $St(2, \mathbb{Z})$ as follows. $SL(2, \mathbb{R})$ is infinitely connected. Call its universal covering group $\widetilde{SL}(2, \mathbb{R})$, so that $\widetilde{SL}(2, \mathbb{R})/\mathbb{Z} = SL(2, \mathbb{R})$. The inverse image of $SL(2, \mathbb{Z})$ in $\widetilde{SL}(2, \mathbb{R})$ is $St(2, \mathbb{Z})$, so that $SL(2, \mathbb{Z}) = St(2, \mathbb{Z})/\mathbb{Z}$. The \mathbb{Z} here corresponds to “ $2\pi N$ -rotations at ∞ ”.

The Steinberg group $St(2, \mathbb{Z})$ can be presented as follows:

$$St(2, \mathbb{Z}) = \langle e_{12}, e_{21}, d : d = (e_{12}e_{21}^{-1}e_{12})^4, de_{ij} = e_{ij}d \rangle. \quad (30)$$

Thus d generates the center \mathcal{C} of $St(2, \mathbb{Z})$. It generates “ 2π -rotations at ∞ ”. The group $SL(2, \mathbb{Z})$ is $St(2, \mathbb{Z})/\mathcal{C}$.

We want to examine the action of $St(2, \mathbb{Z})$ on $\pi_1(\mathbb{R}^2 \# T^2)$. For this, by (30), it is enough to show the action of e_{ij} on the a and b cycles.

In Figure 6, the base point is not at ∞ . So choosing it at ∞ , the above cycles are renamed in Figure 8: the equivalence class of the cycle a is represented by $N \times N$ matrix U_1 and that of the cycle b by the $N \times N$ matrix U_2 .

Under e_{ij} , one has the following³:

$$e_{12}U_1 = U_1, \quad e_{12}U_2 = U_2U_1, \quad (31)$$

$$e_{21}U_1 = U_1U_2, \quad e_{21}U_2 = U_2. \quad (32)$$

As e_{ij} are automorphisms, these equations imply that

$$e_{12}U_1^{-1} = U_1^{-1}, \quad e_{12}U_2^{-1} = U_1^{-1}U_2^{-1}, \quad (33)$$

$$e_{21}U_1^{-1} = U_2^{-1}U_1^{-1}, \quad e_{21}U_2^{-1} = U_2^{-1}. \quad (34)$$

³The details of these results follow from [10].

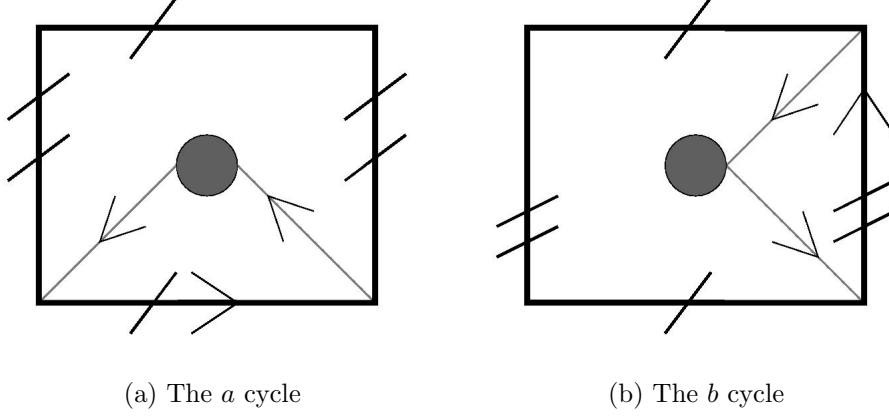


Figure 8: The a and b cycles of $\mathbb{R}^2 \# T^2$ with infinity as base point.

The action on the center c in (25) also follows from the automorphism property:

$$e_{ij}c = e_{ij}(U_1)e_{ij}(U_2)e_{ij}(U_1^{-1})e_{ij}(U_2^{-1}) = c \quad (35)$$

So c is invariant under $St(2, \mathbb{Z})$.

But that is *not* the case with U_1^N and U_2^N . We find

$$e_{12}U_1^N = U_1^N \quad (36)$$

$$e_{12}U_2^N = U_2U_1U_2U_1 \dots U_2U_1 = c^{\frac{N(N-1)}{2}}U_2^N U_1^N = c^{\frac{N(N-1)}{2}}e^{iN\theta_1}e^{iN\theta_2} \quad (37)$$

$$e_{21}U_1^N = U_1U_2U_1U_2 \dots U_1U_2 = c^{\frac{N(N+1)}{2}}U_2^N U_1^N = c^{\frac{N(N+1)}{2}}e^{iN\theta_1}e^{iN\theta_2} \quad (38)$$

$$e_{21}U_2^N = U_2^N. \quad (39)$$

Thus after the diffeo e_{12} , U_1^N has the representation $e_{12}U_1^N = U_1^N = U_1^N$, while U_2^N has the representations $e_{12}U_2^N = U_2^N = e^{\frac{N(N-1)}{2}}e^{iN\theta_1}e^{iN\theta_2}$. Since in general $U_2'^N \neq U_2^N$, e_{12} in general changes the UIRR of $\pi_1(\mathbb{R}^2 \# T^2)$.

Hence the diffeo e_{12} is anomalous for a generic quantization of $\mathbb{R}^2 \# T^2$. Similar remarks are valid for e_{21} .

4.2 Remarks

1. We can be certain that e_{ij} are domain-changing operators and therefore anomalous when they alter the UIRR of $\pi_1(\mathbb{R}^2 \# T^2)$ on which a quantum theory is based.

For $c = 1$, U_i 's are phases. Then e_{ij} are anomalous if they change U_1 or U_2 : that will change the domain.

A simple example is the following:

$$U_1 = e^{i\theta_1}, \quad U_2 = e^{i\theta_2}, \quad c = 1. \quad (40)$$

Then

$$e_{12}U_1 = e^{i\theta_1}, \quad e_{12}U_2 = e^{i(\theta_1+\theta_2)}. \quad (41)$$

So e_{12} is anomalous if $e^{i\theta_1} \neq 1$.

2. Suppose $c \neq 1$ so that $U_1U_2 \neq U_2U_1$, meaning that U_i 's *themselves* are anomalous. That means that we *cannot* implement U_i 's as operators leaving the domain of $i\mathcal{D}$ invariant. That is because they can be thought of as acting by conjugation or adjoint action $\text{Ad } U_i$ on the U_i 's, that is,

$$\text{Ad } U_i(U_j) = U_iU_jU_i^{-1}. \quad (42)$$

But $\text{RHS} \neq U_j$ if $i \neq j$. Thus although the UIRR ρ is invariant, the boundary conditions and hence the domain are not and U_i 's are anomalous.

Another way to say this is as follows. The group $\pi_1(\mathbb{R}^2 \# T^2)$ is a (discrete) *gauge* group. Only *gauge invariant* objects are observables. But if $\pi_1(\mathbb{R}^2 \# T^2)$ has a nonabelian representation ρ in a quantum theory, then only the *center* of $\rho(\pi_1(\mathbb{R}^2 \# T^2))$ commutes with all elements of $\rho(\pi_1(\mathbb{R}^2 \# T^2))$. Only they are observable. This center is in general spanned by c, U_1^N, U_2^N .

These remarks generalize to *any* nonabelian gauge group and is the basis of the proof that nonabelian monopoles break color [24–26].

5 Witten Spinors

There is an aspect of the preceding discussion which requires elaboration. The spatial Dirac (or Laplace) operator is written in a background metric outside the cut-out balls. This metric is flat, the complement of the cut-out ball being the asymptotic region. The entire effect of the mapping class group is treated in terms of its action on domains of $i\mathcal{D}$. It would be attractive to derive an approach where the metric away from the geon is not necessarily flat.

One possibility is to use the Witten spinor ξ or its generalizations [14, 15]. In Witten's original work, they were used to prove the positivity of

the ADM energy in asymptotically flat space-times. The spinor ξ obeys the two-component Dirac equation

$$i\sigma \cdot D\xi = 0, \quad (43)$$

where D_i here comes from the four-dimensional covariant derivative D_μ by restricting μ to spatial indexes. Also ξ approaches a constant ξ_0 (that is to say, a covariantly constant ξ_0 with respect to a flat connection) at ∞ . The physical meaning of ξ_0 is that $\xi_0^\dagger \sigma_\mu \xi_0$, with $\sigma_0 = \mathbb{1}$ and $\sigma_i =$ Pauli matrices, determines a future-pointing null vector. That means that it goes towards fixing a global time direction.

One can also work with Dirac spinors instead,

$$i\gamma \cdot D\xi = 0, \quad (44)$$

with $\xi \rightarrow \xi_0 = a$ constant Dirac spinor as $|\vec{x}| \rightarrow \infty$, so that $\bar{\xi}\gamma\xi$ is a future-pointing time-like vector. There are advantages in using Dirac spinors since they will eliminate the need for distinguishing dotted and undotted spinors.

The positive energy theorem solves (43) and (44) for a specified ξ at ∞ using the Green's function of $i\sigma \cdot D$ or $i\gamma \cdot D$ regarded as a self-adjoint operator with Dirichlet boundary conditions at ∞ . That specifies its domain.

By adding source terms to the Einstein tensor $G_{\mu\nu}$, Witten's proof can be modified to include matter as well. We can also include nonabelian gauge fields in D and carry the proof through after a modification of the dominant energy condition.

Gibbons *et al.* [15] have generalized Witten's proof in another direction by considering black holes and treating the event horizon as a boundary. That involves a choice of boundary conditions at the horizon which is compatible with ellipticity and self-adjointness and is non-trivial.

The point of these remarks is the following. In $(3+1)$ dimensions, we can attach a geon as previously, that is, by carving out a ball B^3 from \mathbb{R}^3 and making identifications. We can then study the operator $i\sigma \cdot D$ or $i\gamma \cdot D$ where the metric need be only asymptotically flat and D can include gauge fields. There is little doubt then that mapping class groups will generically change its domains and will be anomalous. In this approach, the metric away from B^3 is not constrained to be flat: it need be only asymptotically flat.

The positive energy proof however requires that there is a *unique* extension of the boundary data to the bulk. That requires that the Pauli or Dirac operators regarded as self-adjoint operators (with domains specified as in previous sections) have no kernel. We have not tried a proof of this point, which is tied up with issues of ellipticity.

6 On Black Hole Entropy

We continue to focus on $\mathbb{R}^2 \# T^2$.

Although we begin with a domain \mathcal{D}_{U_1, U_2} fixed by the elements U_i of the rational torus, the mapping class group generates an orbit in the space of domains.

Suppose that the dimension of the orbit $D^\infty/D_0^\infty \mathcal{D}_{U_1 U_2}$ is $d(U_1, U_2)$. Then if $|\psi_{U_1, U_2}\rangle\langle\psi_{U_1, U_2}|$ is a state where $|\psi_{U_1, U_2}\rangle \in \mathcal{D}_{U_1, U_2}$, restoration of diffeo invariance requires that we average this state over orbit $D^\infty/D_0^\infty \mathcal{D}_{U_1 U_2}$ and work with

$$\omega = \frac{\Omega}{\text{Tr } \Omega}, \quad (45)$$

where

$$\Omega = \sum_{U'_1, U'_2 \in D^\infty/D_0^\infty \mathcal{D}_{U_1 U_2}} |\psi_{U'_1, U'_2}\rangle\langle\psi_{U'_1, U'_2}|. \quad (46)$$

The state Ω is not pure. Its entropy is very roughly

$$S = -\log d(U_1, U_2). \quad (47)$$

Entropy can thus be generated from attempts at restoration of symmetries.

There are classical singularities theorems of Friedman, Schleich, Witt [27], Gannon [28, 29] showing that if the spatial slice in 3d or 4d is not \mathbb{R}^2 or \mathbb{R}^3 , then the space-time obtained from initial data is geodesically incomplete. This theorem is often interpreted to mean that the above data evolve into black holes. Geons are hence classically expected to evolve into black holes. The entropy S may then contribute to the entropy of the black hole.

This is an interesting possibility, but to make connection to the usual black hole thermodynamics, S must scale with the area of the horizon. We do not obtain this scaling in any obvious way.

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